MATH 3310 Assignment 3

Due: November 5, 2024

1. Consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

   (a) Determine whether the Jacobi method converges.
   (b) Using initial approximation $x^0 = (3, 2, 1)^T$, conduct the first two Jacobi iterations.
   (c) Determine whether the Gauss-Seidel method converges.
   (d) Using initial approximation $x^0 = (2, 1, 0)^T$, conduct the first two Gauss-Seidel iterations.

2. Consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 8 \\ 8 \\ 8 \end{pmatrix}$$

   (a) Determine whether the SOR method converges if $\omega = 1.5$.
   (b) Using initial approximation $x^{(0)} = (6, 4, 0)^T$, conduct the first two SOR iteration.

3. Consider solving $Ax = b$ with the following iterative scheme.

$$x_{k+1} = x_k + \alpha(b - Ax_k)$$

   where $\alpha > 0$. Assume all eigenvalues λ_i of A are real, and such that $\lambda_{min} \leq \lambda_i \leq \lambda_{max}$, $\lambda_{min}, \lambda_{max}$ are the smallest and largest eigenvalues of A, respectively.

   (a) If $\lambda_{min} < 0$ and $\lambda_{max} > 0$, prove that this scheme always diverges for some initial guess.
   (b) Assume $\lambda_{min} > 0$, what are the sufficient and necessary conditions on α for the iterative scheme to converge?

(c) Assume $\lambda_{min} > 0$, what is the best value α_{opt} for α, i.e., the value of α minimizing the convergence factor? And, what is the convergence factor in such case?

4. Consider an $n \times n$ tridiagonal matrix of the form

$$T_\alpha = \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & \alpha \end{pmatrix},$$

where α is a real parameter.

(a) Verify that the eigenvalues of T_α are given by

$$\lambda_j = \alpha - 2\cos(j\theta), \quad j = 1, \ldots, n,$$

where

$$\theta = \frac{\pi}{n+1},$$

and that an eigenvector associated with each λ_j is

$$q_j = [\sin(j\theta), \sin(2j\theta), \ldots, \sin(nj\theta)]^T.$$

Under what condition on α does this matrix become positive definite?

(b) Let $\alpha = 2$.

i. Will the Jacobi iteration converge for this matrix? If so, what will its convergence factor be?

ii. Will the Gauss-Seidel iteration converge for this matrix?

iii. For which values of ω will the SOR iteration converge?

5. Let B be a matrix with the following structure:

$$B = \begin{pmatrix} O & B_{12} \\ B_{21} & O \end{pmatrix},$$

and let L and U be the lower and upper triangular parts of B, respectively. Two zero matrices in the diagonal are squared.

(a) If μ is an eigenvalue of B, then so is $-\mu$.

(b) The eigenvalues of the matrix

$$B(\alpha) = \alpha L + \frac{1}{\alpha} U$$

defined for $\alpha \neq 0$ are independent of α.